Form Factors of Few-Body Systems:
Point Form Versus Front Form

M. Gómez-Rocha, E. P. Biernat, W. Schweiger

Abstract We present a relativistic point-form approach for the calculation of electroweak form factors of few-body bound states that leads to results which resemble those obtained within the covariant light-front formalism of Carbonell et al. Our starting points are the physical processes in which such form factors are measured, i.e. electron scattering off the bound state, or the semileptonic weak decay of the bound state. These processes are treated by means of a coupled-channel framework for a Bakamjian-Thomas type mass operator. A current with the correct covariance properties is then derived from the pertinent leading-order electroweak scattering or decay amplitude. As it turns out, the electromagnetic current is affected by unphysical contributions which can be traced back to wrong cluster properties inherent in the Bakamjian-Thomas construction. These spurious contributions, however, can be separated uniquely, as in the covariant light-front approach. In this way we end up with form factors which agree with those obtained from the covariant light-front approach. As an example we will present results for electroweak form factors of heavy-light systems and discuss the heavy-quark limit which leads to the famous Isgur-Wise function.

Keywords Point-form dynamics · Relativistic quantum mechanics · Hadron structure

1 Introduction

Electroweak processes in which either an electron is scattered elastically off a hadron or the hadron decays weakly into another hadron and an electron-antineutrino pair provide a lot of information about the composition of hadrons in terms of their constituents. The electric and weak coupling constants are small enough such that leading-order perturbation theory suffices to get meaningful results for the scattering and decay

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probabilities. In leading-order the invariant scattering or decay amplitudes become just contractions of a leptonic with a hadronic current (times a γ, or W propagator). The most general covariant decomposition of the hadronic current leads to the introduction of form factors. These are Lorentz-invariant functions of the 4-momentum transferred to the hadron. They contain all the information about the substructure of the hadron, i.e. the deviation from a point-like hadron, and can be directly extracted from (polarized) scattering or decay cross sections.

The theoretical challenge is now to relate the hadron current to the currents of its constituents. Under Poincaré transformations a hadronic current operator should transform covariantly [1]. Since the binding interaction shows up in some of the Poincaré generators, depending on the form of relativistic dynamics to be used, it follows that $\hat{J}^\mu(x)$ must, in general, also depend on the binding interaction and cannot be a simple sum of the constituent currents. Further constraints for a theoretical model of a hadron current come from current conservation, i.e. $\partial_\mu \hat{J}^\mu(x) = 0$, and from the requirement that the hadron charge should be the sum of the constituent charges, independent on whether the binding interaction is present or not.

We are primarily interested in calculating electroweak form factors of strongly bound few-body systems within the framework of relativistic quantum mechanics. A common procedure is to calculate the bound-state wave function for a given binding force and use it to *construct* a model for only the minimum number of current components that is needed to fix the form factors uniquely. The remaining current components are then determined by covariance and current conservation. In usual front-form, e.g., it suffices to know the $\hat{J}^+$ component if one calculates the current in the $q^+ = q^- = 0$ frame [1]. This kind of procedure, however, has the drawback that the results for the form factors will, in general, slightly depend on the chosen current components and on the reference frame in which the construction of the current is done [2]. Our strategy is rather to *derive* a full 4-vector current that is compatible with the binding forces and valid in any reference frame within a Poincaré invariant quantum mechanical setting. As it turns out, such a current exhibits some unphysical features which, however, can be split off in a unique way leaving a 4-vector current with all the desired properties. Surprisingly, the outcome of our approach resembles very much the results obtained within the covariant light-front formalism that was suggested in Ref. [2]. This is the more remarkable since we use the point form of relativistic quantum mechanics. In this form all components of the 4-momentum become interaction dependent, whereas the Lorentz generators stay free of interactions. This guarantees simple boost and covariance properties of wave functions and physical observables, respectively.

2 Relativistic multichannel formalism and hadron currents

Our derivation of electroweak form factors starts with the physical processes in which the form factors are measured, i.e. elastic electron-hadron scattering and the weak decay of hadrons. We describe these reactions by means of a coupled-channel framework in which the dynamics of the intermediate gauge bosons – either a photon or a W-boson – is fully taken into account. Poincaré invariance is ensured by employing the Bakamjian-Thomas construction [3]. In its point-form version the (interacting) 4-momentum operator $\hat{P}^\mu$ is factorized into an interacting mass operator and a free 4-velocity operator $\hat{P}^\mu = \hat{M}\hat{V}_{\text{free}}^\mu$. It is therefore only necessary to study an eigenvalue problem for the mass operator.

We will exemplify our approach through elastic electron-meson scattering with the meson being a spin-0 confined quark-antiquark system. In this case a mass eigenstate $\hat{M}|\psi\rangle = m|\psi\rangle$ is written as a direct sum of a quark-antiquark-electron component $|\psi_{q\bar{q}e}\rangle$ and a quark-antiquark-electron-photon component $|\psi_{q\bar{q}e\gamma}\rangle$. The mass eigenvalue equation to be solved has then the form

$$
\begin{pmatrix} \hat{M}_{q\bar{q}e} & \hat{K}_{\mathrm{em}} \\ \hat{K}_{\mathrm{em}}^{\dagger} & \hat{M}_{q\bar{q}e\gamma} \end{pmatrix} \begin{pmatrix} |\psi_{q\bar{q}e}\rangle \\ |\psi_{q\bar{q}e\gamma}\rangle \end{pmatrix} = m \begin{pmatrix} |\psi_{q\bar{q}e}\rangle \\ |\psi_{q\bar{q}e\gamma}\rangle \end{pmatrix} ,
\tag{1}
$$

where $M_{q\bar{q}e}$ and $M_{q\bar{q}e\gamma}$ consist of a kinetic term and an instantaneous confining potential between quark and antiquark, and $\hat{K}_{\mathrm{em}}^{(\dagger)}$ is a vertex operator which accounts for the absorption (emission) of a photon by the electron or (anti)quark. It is derived from the interaction Lagrangean density of QED [4].

The current matrix elements that are necessary for the calculation of the electromagnetic meson form factors can be extracted from the invariant 1-photon-exchange amplitude. This is essentially given by the on-shell matrix elements of the optical potential $\hat{V}_{\mathrm{opt}}(m) := \hat{K}_{\mathrm{em}}(\hat{M}_{q\bar{q}e\gamma} - m)^{-1}\hat{K}_{\mathrm{em}}^{\dagger}$. These matrix elements exhibit the expected structure:

$$
\mathcal{M}_{1\gamma}(\mathbf{k}_e', \mu_e'; \mathbf{k}_e, \mu_e) \propto \langle V'; \mathbf{k}_e', \mu_e'; \mathbf{k}_M' | \hat{V}_{\mathrm{opt}}(m) | V; \mathbf{k}_e, \mu_e; \mathbf{k}_M \rangle_{\mathrm{on-shell}}
$$
$$
\propto V^0 \delta^3(\mathbf{V} - \mathbf{V}') \frac{j_{\mathrm{em}\,\mu}(\mathbf{k}_e', \mu_e'; \mathbf{k}_e, \mu_e) J_{\mathrm{em}}^{\mu}(\mathbf{k}_M'; \mathbf{k}_M)}{(k_e' - k_e)^2}
\tag{2}
$$

such that the meson current $J_{\mathrm{em}}^{\mu}(\mathbf{k}_M'; \mathbf{k}_M)$ can be easily identified. $|V^{(\prime)}; \mathbf{k}_e^{(\prime)}, \mu_e^{(\prime)}; \mathbf{k}_M^{(\prime)}\rangle$ are, so called, "velocity states" that specify the state of a system by its overall velocity $V^{(\prime)}$, the center-of-mass momenta $\mathbf{k}_i^{(\prime)}$ and the canonical spins $\mu_i^{(\prime)}$ of its components [5]. In our case $\mathbf{k}_M^{(\prime)}$ is the momentum of the confined q-$\bar{q}$ subsystem with the quantum numbers of the meson. "On-shell" means that $m = k_e^0 + k_M^0 = k_e'^0 + k_M'^0$ and $k_e^0 = k_e'^0$, $k_M^0 = k_M'^0$. A detailed derivation of Eq. (2) and the explicit expression for the meson current $J_{\mathrm{em}}^{\mu}(\mathbf{k}_M'; \mathbf{k}_M)$ in terms of the constituent currents and the bound-state wave functions are given in Refs. [6,7,8].

It is quite obvious, how this formalism can be generalized to obtain an expression for the weak meson transition current $J_{\mathrm{wk}}^{\mu}(\mathbf{k}_{M'}'; \mathbf{k}_M)$ that enters the semileptonic $M \to M' e^- \bar{\nu}_e$ decay. The leading-order invariant transition amplitude $\mathcal{M}_{1W}$ in this case can be derived from a 4-channel problem. In addition to the incoming $q\bar{q}$ channel and the outgoing $q'\bar{q}e\bar{\nu}_e$ channel one needs a $q'\bar{q}W$ and a $q\bar{q}We\bar{\nu}_e$ channel to account for the intermediate states in which the W-boson is in flight. Here we assume that the flavor change due to the coupling of the W-boson happens for the quark. $\mathcal{M}_{1W}$ is again given by on-shell matrix elements ($m = m_M = k_M^0 = k_{M'}^{0\prime} + k_e^{0\prime} + k_{\bar{\nu}_e}^{0\prime}$) of the optical transition potential $\hat{V}_{\mathrm{opt}}^{M \to M'}(m) := \hat{K}_{\mathrm{wk}}(\hat{M}_{q'\bar{q}W} - m)^{-1}\hat{K}_{\mathrm{wk}}^{\dagger} + \hat{K}_{\mathrm{wk}}(\hat{M}_{q\bar{q}We\bar{\nu}_e} - m)^{-1}\hat{K}_{\mathrm{wk}}^{\dagger}$. The vertex operators $K_{\mathrm{wk}}^{(\dagger)}$ for the absorption (emission) of the W by the quarks and leptons are derived from the interaction Lagrangean density of QFD [4]. The mass operators $M_{q'\bar{q}W}$ and $M_{q\bar{q}We\bar{\nu}_e}$ contain again an instantaneous confining potential between quark and antiquark. The invariant transition amplitude $\mathcal{M}_{1W}$ has the same structure as the 1-photon-exchange amplitude $\mathcal{M}_{1\gamma}$ (cf. Eq. (2)) with the electromagnetic currents replaced by the weak currents and the photon propagator $(k_e' - k_e)^{-2}$ replaced by the (covariant) W-propagator $((k_e' + k_{\bar{\nu}_e}')^2 - m_W^2)^{-1}$. A detailed derivation and the explicit expression for the weak meson transition current $J_{\mathrm{wk}}^{\mu}(\mathbf{k}_{M'}'; \mathbf{k}_M)$ in terms of the quark current and the bound-state wave functions can be found in Ref. [8].
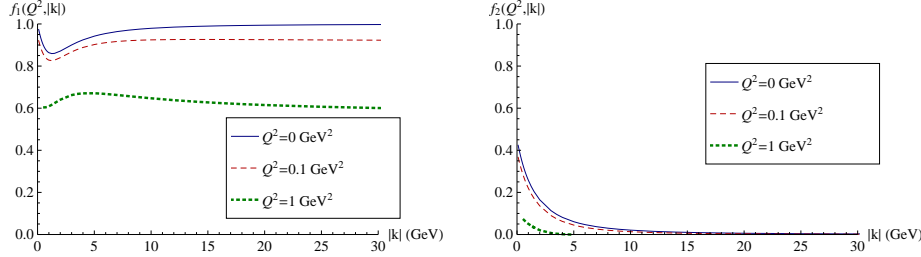
Fig. 1 Dependence of the electric D-meson form factor f_1 and the spurious D-meson form factor f_2 (cf. Eq.(4)) on the meson center-of-mass momentum $k = |\mathbf{k}_M|$ for different values of the momentum transfer Q^2.

3 Electromagnetic current and form factors

As a next step we will analyze the properties of the electromagnetic meson current $J_{\mathrm{em}}^\mu(\mathbf{k}'_M; \mathbf{k}_M)$ that follows from Eq. (2). Since we are using velocity states in which $\mathbf{k}_M$ and $\mathbf{k}'_M$ are always defined in the center-of-mass of the electron-meson system, $J_{\mathrm{em}}^\mu(\mathbf{k}'_M; \mathbf{k}_M)$ does not transform like a 4-vector under a Lorentz transformation Λ, but it rather transforms by the Wigner rotation $R_W(V, \Lambda)$. A current with the correct transformation properties is obtained by going back to the physical meson momenta $p_M^{(\prime)} = B_c(V) k_M^{(\prime)}$, i.e. by boosting the center-of-mass momenta with the overall velocity V of the electron-meson system:

$$J_{\mathrm{em}}^\mu(\mathbf{p}'_M; \mathbf{p}_M) := [B_c(V)]^\mu_\nu J_{\mathrm{em}}^\nu(\mathbf{k}'_M; \mathbf{k}_M). \tag{3}$$

$B_c(V)$ is the boost matrix of a rotationless boost. If M is a pseudoscalar meson it can be shown that $J_{\mathrm{em}}^\mu(\mathbf{p}'_M; \mathbf{p}_M)$ is conserved, i.e. $(p'_M - p_M)_\mu J_{\mathrm{em}}^\mu(\mathbf{p}'_M; \mathbf{p}_M) = 0$ [6,7]. Since internal momenta are integrated over, the most general covariant decomposition of J_{em}^μ thus takes on the form

$$J_{\mathrm{em}}^\mu(\mathbf{p}'_M; \mathbf{p}_M) = (p_M + p'_M)^\mu f_1(Q^2, s) + (p_e + p'_e)^\mu f_2(Q^2, s), \tag{4}$$

with $Q^2 = -(p'_M - p_M)^2$ and $s = (p_M + p_e)^2$. With the Bakamjian-Thomas construction we have a nice Poincaré invariant treatment of the electron-meson system, but the price we pay is a dependence of the meson current on the electron momenta which should not be there. It is the consequence of wrong cluster properties, a well known drawback of the Bakamjian-Thomas construction [1]. As numerical studies reveal, however, the dependence on the electron momenta becomes negligible if the invariant mass $\sqrt{s}$ of the electron-meson system is taken large enough [6,7]. This is demonstrated in Fig. 1 for the case of a D^+ meson and a simple Gaussian $\psi(\kappa) \propto \exp(-\kappa^2/(2a^2))$, $a = 0.55$ GeV taken for the bound-state wave function (the quark masses are $m_c = 1.6$ GeV, $m_{u,d} = 0.25$ GeV).

In order to get rid of the spurious $k_e^{(\prime)}$ dependencies it is thus tempting to take the limit $s \to \infty$. In this limit the electromagnetic current of a pseudoscalar meson acquires indeed its usual form $J_{\mathrm{em}}^\mu(\mathbf{k}'_M; \mathbf{k}_M) = (k'_M + k_M)^\mu F(Q^2)$ and the analytical expression for the electric form factor becomes rather simple. For equal quark and antiquark masses and pure s-wave pseudoscalar mesons it is given by [6,7]

$$F(Q^2) = \lim_{s \to \infty} f_1(Q^2, s) = \frac{1}{4\pi} \int \mathrm{d}^3 \tilde{k}' \sqrt{\frac{\tilde{k}^0}{\tilde{k}^{0\prime}}} \, \mathcal{S} \, \psi^*(|\tilde{\mathbf{k}}'|) \psi(|\tilde{\mathbf{k}}|), \tag{5}$$
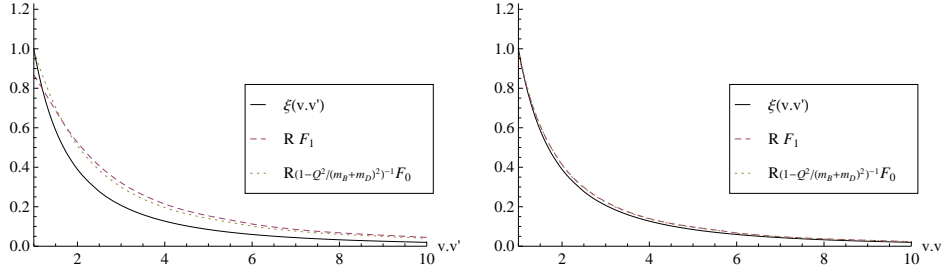
Fig. 2 Weak $B^- \to D^0 e^- \bar{\nu}_e$ decay form factors (multiplied with appropriate kinematical factors) for finite heavy-quark masses in comparison with the Isgur-Wise function. Results for the physical b- and c-quark masses are shown in the left figure. In the right figure these masses are multiplied by a factor 10.

with the spin-rotation factor $\mathcal{S}$ being the trace of a product of Wigner D functions. Primed and unprimed momenta are related by appropriate rotationless boosts, $\tilde{k} = \lim_{s\to\infty} B_c^{-1}(v_{q\bar{q}})[B_c(v'_{q\bar{q}})\tilde{k}' + k_M - k'_M]$. Remarkably, by a simple change of integration variables the form factor expression in Eq. (5) goes over into the standard front-form result for a spectator current in the $q^+ = 0$ frame [6,7] with $\mathcal{S}$ becoming the Melosh-rotation factor.

The generalization to pseudoscalar mesons with unequal-mass constituents is straightforward [8]. What is interesting in this connection is the heavy-quark limit (HQL) in which the mass of the heavy constituent (say the quark) and hence also of the meson goes to infinity. This limit has to be taken in such a way that $v_M \cdot v'_M = p_M \cdot p'_M / m_M^2$ stays constant and $m_q = m_M$. In this limit the spurious contributions are also seen to vanish and one finds that [8]

$$J_{\text{em}}^\mu(\mathbf{p}'_M; \mathbf{p}_M) \xrightarrow{HQL} m_M (v_M + v'_M) \xi(v_M \cdot v'_M) \tag{6}$$

with the Isgur-Wise function [9]

$$\xi(v_M \cdot v'_M) = \frac{1}{4\pi} \int \mathrm{d}^3 \tilde{k}'_{\bar{q}} \sqrt{\frac{\tilde{k}_{\bar{q}}}{\tilde{k}_{\bar{q}}^{0\prime}}} \sqrt{\frac{2}{1 + v_M \cdot v'_M}} \, \mathcal{W} \, \psi^*(|\tilde{\mathbf{k}}'_{\bar{q}}|) \, \psi(|\tilde{\mathbf{k}}_{\bar{q}}|) \,. \tag{7}$$

The Wigner rotation factor $\mathcal{W}$ is a function of $\tilde{k}'_{\bar{q}}$ and $(v_M \cdot v'_M)$. Fig. 2 shows the result for the Isgur-Wise function obtained with the same (light) quark mass and the same Gaussian bound-state wave function as in Fig. 1. This input has also been used in a front-form calculation of the Isgur-Wise function [10] and we find indeed numerical agreement with the result of Cheng et al.

4 Weak current and form factors

Let us next turn to the analysis of the weak current J_{wk}^μ that can be extracted from the semileptonic decay amplitude $\mathcal{M}_{1W}$. We will again consider a pseudoscalar to pseudoscalar transition. As in the electromagnetic case (cf. Eq. (3)) we have to go back to the physical particle momenta to obtain a current that transforms like a 4-vector. We have to note, however, that the momentum transferred to the meson is now timelike,

whereas it is spacelike in electron-meson scattering. Interestingly, wrong cluster properties of the Bakamjian-Thomas construction do not entail unphysical properties of the decay current. Therefore its covariant decomposition takes on the usual form [11]:

$$
J_{\mathrm{wk}}^{\mu}(\mathbf{p}'_{M'}; \mathbf{p}_M) = \left((p_M + p'_{M'})^{\mu} - \frac{m_M^2 - m_{M'}^2}{q^2} \right) F_1(q^2) + \frac{m_M^2 - m_{M'}^2}{q^2} q^{\mu} F_0(q^2) ,
\tag{8}
$$

with $q = (p_M - p'_{M'})$. If heavy-quark (flavor) symmetry holds the heavy-quark limit of F_0 and F_1 (multiplied with appropriate kinematical factors) should give the same (universal) Isgur-Wise function as the heavy-quark limit of the electromagnetic form factor (q^2 again replaced by $(v'_{M'} \cdot v_M)$). This is indeed the case, which proves that our procedure to calculate currents and form factors respects heavy-quark symmetry. But we can also calculate the form factors for finite (physical) quark masses to estimate how far nature is away from the heavy quark limit. The result for the $B^- \to D^0 e^- \bar{\nu}_e$ decay is plotted in Fig. 2. For physical heavy-quark masses the deviation is sizable. Approximate restauration of heavy-quark symmetry is, however, observed for masses that are about 10 times larger.

5 Concluding remarks

We have seen for spin-0 2-particle bound states and instantaneous binding forces that our point-form approach provides results for electromagnetic form factors that agree with front form calculations in the $q^+ = 0$ frame. We agree also in the spin-1 case with the outcome of the covariant front-form approach of Carbonell et al. [2]. This is discussed elsewhere [7]. We have further calculated weak decay form factors for heavy-light systems. Our formalism is seen to give the right heavy-quark limit with the Isgur-Wise function being again in agreement with the front form result. This has also been checked for pseudoscalar to vector transitions [8], which verifies heavy-quark spin symmetry. What remains to be seen is, whether the agreement with form-factor calculations in front form will continue to hold for binding forces caused by dynamical particle exchanges.

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